Exercise 19

Prove the identity.

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$$

(n any real number).

Solution

Work out the left side using the definitions listed on page 259.

$$\begin{aligned}
\cosh nx + \sinh nx &= \left(\frac{e^{nx} + e^{-nx}}{2}\right) + \left(\frac{e^{nx} - e^{-nx}}{2}\right) \\
&= \frac{1}{2}e^{nx} + \frac{1}{2}e^{-nx} + \frac{1}{2}e^{nx} - \frac{1}{2}e^{-nx} \\
&= e^{nx}
\end{aligned}$$

Work out the right side as well.

$$\begin{aligned}
(\cosh x + \sinh x)^n &= \left(\frac{e^{x} + e^{-x}}{2} + \frac{e^{x} - e^{-x}}{2}\right)^n \\
&= \left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x} + \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}\right)^n \\
&= (e^x)^n \\
&= e^{nx}
\end{aligned}$$

Therefore,

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx,$$

where n is any real number.